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MULTI-SINGULAR AND MULTI-AFFINE PROPERTIES OF BOUNDED CASCADE MODELS

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Abstract

We investigate a class of one-dimensional bounded random cascade models which are multiplicative and stationary by construction but additive and non-stationary with respect to some, but not all, of their statistical properties. In essence, a new parameter $H > 0$ is introduced to "smooth" standard (p-model) cascades, these well-studied processes being retrieved at $H = 0$. The resulting ambivalent statistical behavior of the new model leads to a 1st order multifractal phase transition in the structure function exponents, i.e., there is a discontinuity in the derivative of $\zeta_q = \min\{qH, 1\}$. We interpret this bifurcation as a separation of the stationary and non-stationary "ingredients" of the model by lowering the multifractal "temperature" $(1/q)$ below the critical value H. We also see exactly how the generalized dimensions D_q converge to one in the small scale limit for all q. We discuss this last finding in terms of "residual" multifractality, a singularity spectrum that is entirely traceable to finite size effects (to which we are never immune in data analysis situations). Finally, we locate the bounded and unbounded versions of the model in the "$q = 1$ multifractal plane" where the coordinates are $C_1 = 1 - D_1$ (a direct measure of "intermittency") and $H_1 = \zeta_1$ (a direct measure of "smoothness"), both of which are normally in the interval $[0, 1]$. This provides us with a simple way of comparing the multiplicative models with their additive counterparts, as well as with different types of geophysical data.

1. MOTIVATION AND OVERVIEW

Stationarity is a constraint that considerably limits the direct applicability of scale-invariant stochastic models since, on the one hand, the spectral exponent β does not exceed one in this case[1] whereas, on the other hand, natural signals or fields often have $\beta > 1$. Turbulent velocity[2] and passive scalar[3,4] fields come to mind ($\beta = 5/3$), but many other geophysical signals are also in this category.[5,6] In the following section, we describe a class of models with non-stationary small scale behavior without leaving the general framework of multiplicative cascades. Two different multifractal statistics are then used to characterize them: singular measures (Sec. 3) and structure functions (Sec. 4) which yield respectively the exponent hierarchies D_q and H_q. Two interesting concepts are also discussed: respectively, "residual" multifractality and "multi-affine"[7] processes that undergo 1st order multifractal phase transitions. Finally, in Sec. 5, we summarize our findings using the "$q = 1$ multifractal plane"[1]; its coordinates are $C_1 = 1 - D_1$ and H_1 which quantify respectively the degrees of inhomogeneity (in the sense of intermittency) and of non-stationarity (in the sense of signal smoothness) in the system.

2. BOUNDED AND SINGULAR MULTIPLICATIVE CASCADE MODELS

Consider a homogeneous distribution of some substance on the unit interval. Let $\phi_n(x_m)$ be mth part of the substance distribution after nth step. First ($n = 1$), we divide the unit interval into λ_0 equal parts of length $1/\lambda_0$ where λ_0 is an interger > 1, and then weights W_{ij}, $j = 1, \ldots, \lambda_0$, are applied multiplicatively. In the next step ($n = 2$), each sub-interval is further divided into λ_0 equal parts with corresponding random weights $W_{2j} \geq 0$, and so on. On the nth step, the distribution of the original substance would be:

$$\phi_{n+1}(x_{\lambda_0 m - j+1}) = \phi_n(x_m) W_{nj}\,, \quad j = 1, \ldots, \lambda_0\,, \quad m = 1, \ldots, \Lambda_n\,, \quad \phi_0(x_1) = 1\,, \tag{1a}$$

where $\Lambda_n = \lambda_0^n$ ($n = 1, 2, \ldots$) is the grid size in pixels. Then

$$\phi_n(x_m) = \prod_{i=1}^{n} W_{ij}\,, \quad j = 1, \ldots, \lambda_0\,, \quad m = 1, \ldots, \Lambda_n\,, \quad n = 1, 2, \ldots\,. \tag{1b}$$

When Λ_n is large enough, the above processes can be highly intermittent and are generally referred to as "multifractals", emphasizing the multiplicative modulation and the sparseness of the sets where the substance is ultimately concentrated. If we let $\lambda_0 = 2$ and, independently of $i = 1, \ldots, n$, take W_{i1} to be either $2p$ ($0 \leq p \leq 1/2$) or $2(1-p)$ with equal probability, and $W_{i2} = 1 - W_{i1}$, then we retrieve a cascade model known in the turbulence literature as a "p-model".[8] We know that, for such standard (W_{ij} distributed independently of i) conservative ($\langle W \rangle = 1$) cascades, the energy spectrum scales as $k^{-\beta}$ with[9]:

$$\beta = 1 - \ln_{\lambda_0} \langle W^2 \rangle < 1\,, \tag{2}$$

the inequality following directly from Schwartz's, $\langle W^2 \rangle > \langle W \rangle^2 = 1$.

Now consider Eq. (1) again with $\lambda_0 = 2$ and $W_{ij} = 1 \pm f_i$, ($0 \leq f_i \leq 1$, $j = 1, 2$) randomly but this time with $f_i \to 0$ as $i \to \infty$. These models have upper and lower bounds.[10] We will further assume an algebraic decay with (inner) scale $r_i = 1/\Lambda_i = 2^{-i}$:

$$f_i = (1-2p)r_i^H\,, \quad 0 \le p \le \frac{1}{2}\,, \quad 0 \le H < \infty\,, \tag{3}$$

as originally suggested in Ref. 11. The choice $p = 1/2$ yields homogeneous fields for any H. If $H = 0$, the model reverts to a p-model. If furthermore $p = 0$ we find δ-functions positioned at random in $[0,1]$; one of the two sub-eddies is made "dead" and the other remains "alive", starting randomly with right or left. The parameter H clearly produces a radical smoothing that converts a cascade of the usual (singular) kind into a more tame bounded field. The limit $H \to \infty$ formally leads to a Heaviside step (at $x = 0.5$) for any $p \neq 1/2$. See Ref. 10 for a discussion of the scaling regimes.

3. SINGULAR MEASURES

Consider the one-dimensional process $\varphi(x) \ge 0$, $x \in [0,\,1]$, and the associated measure:

$$p_r(x) = \frac{\displaystyle\int_x^{x+r} \varphi(x')\,dx'}{\displaystyle\int_0^1 \varphi(x')\,dx' \ge 0}\,, \quad x \in [0,\,1-r]\,. \tag{4}$$

If $\varphi(x)$ is stationary, we have $\langle p_r(x)^q\rangle = \langle p_r^q\rangle$ for any q, independently of x. If furthermore $\varphi(x)$ is scale-invariant, we can posit:

$$\langle p_r^q\rangle \sim r^{1+(q-1)D_q}\,, \quad 0 < r \ll 1\,, \tag{5}$$

where the generalized dimensions D_q appear. In homogeneous situations, i.e., $\langle p_r^q\rangle \approx \langle p_r\rangle^q \approx r^q$ (for all r and q), Eq. (5) yields $D_q \equiv 1$ (we assume in the following that $\varphi(x) \neq 0$ almost surely everywhere). In contrast, $D_q \neq 1$ for $q \neq 0$ translates to very skewed intermittent $\varphi(x)$'s. We know that D_q is a non-increasing function of q.[12,13] Thus D_1 (the "information" dimension) is generally smaller than $D_0 = 1$ and the codimension $C_1 = 1 - D_1$ is a straightforward measure of inhomogeneity in the system.

Applying singularity analysis to the above model, Eqs. (4) and (5) lead to:

$$D_q = \lim_{n\to\infty} D_q^{(n)} = \lim_{n\to\infty} \frac{1}{q-1}\left\{q - \frac{1}{n}\sum_{i=1}^{n} \ln_2[(1+f_i)^q + (1-f_i)^q]\right\}. \tag{6}$$

From Eq. (6), we can find the following measure of intermittency $C_1^{(n)} = 1 - D_1^{(n)}$ at each cascade level n. In the bounded case, we have $C_1^{(n)} \to C_1 \equiv 0$ $(n \to \infty)$ as long as $\sum f_i^2 < \infty$ and in fact $D_q^{(n)} \to D_q \equiv 1$ $(n \to \infty)$ for all q. The Legendre transform of $(q-1)D_q$ yields the singularity spectrum $f(\alpha)$[14] the $f(\alpha)$ associated with Eq. (6) of course converges to a single point. For technical details, we refer to Ref. 10 where analytical estimates of convergence rates are obtained for the specific case in Eq. (3).

For the intermittent ($H = 0$) processes, the singularities are not smoothed with the increase of n and neither D_q nor $f(\alpha)$ depends on n. In this limit the model is unbounded (identical to the p-model) and has a non-trivial singularity spectrum. The bounded ($H > 0$) model is no longer a "multifractal" in the sense of singular measures ($D_q \equiv 1$, $\alpha = f(\alpha) = 1$)

in spite of its intrinsically multiplicative character. However, in this case there always remains a "residual" or "spurious" multifractality at finite n. In data analysis applications (always at finite resolution), this cannot be distinguished from true multifractality with a sufficiently narrow singularity spectrum.

4. STRUCTURE FUNCTIONS

The generic stochastic process $\varphi(x)$ is now allowed to take any real value and we define:

$$\Delta\varphi(r;\, x) = \varphi(x+r) - \varphi(x)\,, \quad x \in [0,\, 1-r]\,.$$

If $\varphi(x)$ has "stationary increments", then the structure functions $\langle|\Delta\varphi(r;\, x)|^q\rangle = \langle|\Delta\varphi(r)|^q\rangle$ will be independent of x. For scaling processes, we anticipate:

$$\langle|\Delta\varphi(r)|^q\rangle \sim r^{\zeta_q}\,, \quad 0 < r \ll 1\,, \tag{7}$$

where ζ_q is concave on $(-1, \infty)$ and non-decreasing, at least for bounded processes.[10,15] As a counterpart to the exponent hierarchy D_q, we have $H_q = \zeta_q/q$ which, given that $\zeta_0 = 0$, is non-increasing for any ζ_q. As an analog to the singularity spectrum $f(\alpha)$, one can define $D(h)$ by taking the Legendre transform of $\zeta_q + 1$.[16] The multifractal interpretation of $D(h)$ is the fractal dimension of the subset of $[0, 1]$ where the local Hölder exponent $h(x) = h$; this last exponent is defined in $|\Delta\varphi(r;\, x)| \sim r^{h(x)}$. Scaling stationary processes are retrieved at $\zeta_q \equiv 0$ ($h = 0$, $D(h) = 1$) and as long as non-stationary prevails, $\zeta_q > 0$ for $q > 0$, we have in particular[9]

$$\beta = \zeta_2 + 1 = 2H_2 + 1 > 1\,. \tag{8}$$

Narrowly distributed increments, $\langle|\Delta\varphi(r)|^q\rangle \approx \langle|\Delta\varphi(r)|\rangle^q$, lead to $\zeta_q = q\zeta_1$, i.e., $H_q \equiv$ constant. We thus retrieve fractional Brownian motions (fBm's) which generalize standard Brownian motion ($H_q \equiv 1/2$); these are probably the best known examples of "self-affine" random processes.[17] Now statistical self-affinity is characterized by the existence of an exponent $H \in (0,\, 1)$ such that $\langle|\Delta\varphi(\lambda r)|\rangle \sim \lambda^H\langle|\Delta\varphi(r)|\rangle$, $\lambda > 0$, from our standpoint a statement at $q = 1$ only. In contrast, Eq. (7) reads as $\langle|\Delta\varphi(\lambda r)|^q\rangle \sim \lambda^{q^{H_q}}\langle|\Delta\varphi(r)|^q\rangle$ and processes with non-constant H_q are called "multi-affine".[7] In other words, the $q = 1$ case can be related to the fractal dimension of the graph $g(\varphi)$ of $\varphi(x)$, viewed as an object in two-dimensional space[17]:

$$D_g = 2 - \zeta_1 = 2 - H_1 < 2\,. \tag{9}$$

It is noteworthy that $H_1 = \zeta_1 > 0$ in Eq. (7) is equivalent to a statement of stochastic continuity, i.e., increments over small distances are (almost surely everywhere) small. We can view $H_1 = 2 - D_g$, the codimension of $g(\varphi)$, as a direct measure of "smoothness" in the signal $\varphi(x)$ which in turn is related to the degree of non-stationarity. Recall that fBm's with $H < 1/2$ is referred to as "anti-persistent" (successive increments anti-correlate, the process is more stationary) while their counterparts with $H > 1/2$ are "persistent" (successive increments correlate, the process is less stationary).

For the model in Eq. (3), one finds[10]:

$$\zeta_q = qH_q = \min\{qH,\, 1\}\,, \quad 0 \le q \le \infty\,, \tag{10}$$

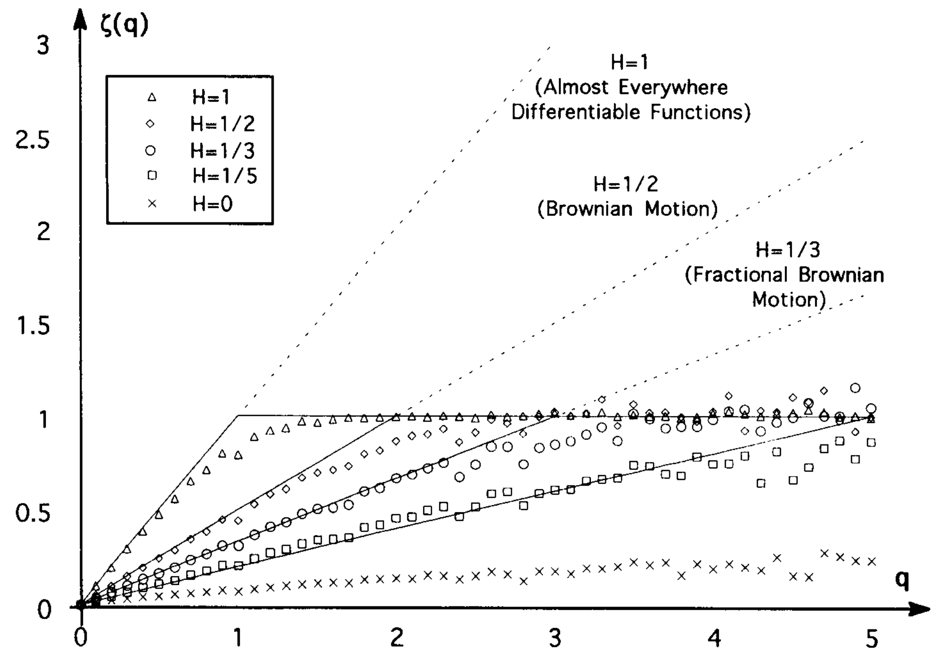

Fig. 1 The structure functions exponents of bounded cascade models for $H = 0, 1/3, 1/2, 1$. The exponents were calculated theoretically (full lines) and numerically (symbols); in the latter case, each q corresponds to a single but different realization, hence the scatter. The corresponding $D(h)$ spectrum consists of only two points: $D(h = H) = 1$ (i.e., a space-filling set) and $D(h = 0) = 0$ (i.e., a finite number of points). The dotted lines illustrate the behavior of ζ_q for additive processes (fBm's) with H equal to 1 (the limiting case of almost everywhere differentiable functions), 1/2 (standard Brownian motion) and 1/3 (a typical "anti-persistent"[17] process). Notice that this last case yields a "Kolmogorov" energy spectrum ($\beta = 5/3$) and a "critical" exponent $q_c = 1/H = 3$. The theory predicts trivial structure functions ($\zeta_q \equiv 0$) for singular models ($H = 0$) since they are stationary; the numerics are representative of finite size effects ("residual" multi-affinity).

a linear behavior like (mono-affine) fBm's for $0 \leq q \leq 1/H$; and $\zeta_q = 1$ for $q > 1/H$. In Fig. 1, we show computationally and theoretically obtained scaling exponents ζ_q versus q for different values of H. The unbounded (p-model) limit at $H = 0$ is quite interesting. It shows a "residual" shift from $\zeta_q \equiv 0$ in the same sense as discussed in connection with D_q but for multi-affinity, i.e., it is entirely traceable to the finite number of cascade levels ($n = 15$ in this case). Three moments deserve to be considered in more detail.

$q = 1$. From Eqs. (9) and (10), we find

$$D_g = 2 - H_1 = \max\{2 - H, 1\}. \tag{11}$$

If $H > 1$, the graphs of bounded cascade fields are statistically indistinguishable from "simple" curves with dimension 1; they in fact closely resemble piece-wise constant functions.[10] In the opposite (p-model) limit $H = 0$, the model is intermittent and stationary and its graph dimension is 2 (independently of $p < 1/2$). Alternatively, bounded models are stochastically continuous ($H_1 > 0$) but nowhere differentiable (as long as $H < 1$), unbounded ones are discontinuous ($H_1 = H = 0$), unless we are dealing with the degenerate case of flat fields ($p = 1/2$).

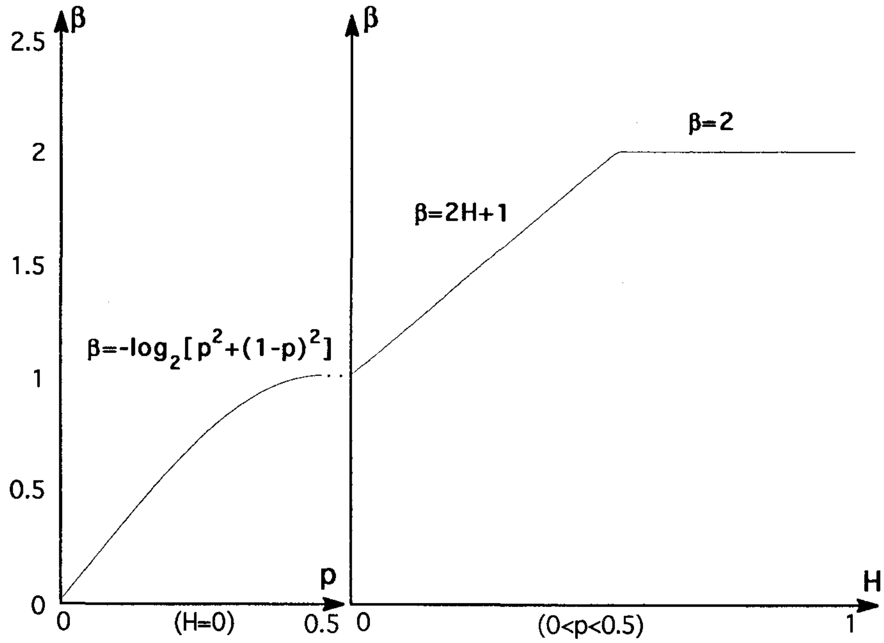

Fig. 2 Schematical plot of the spectral exponent β of bounded and singular cascade models. Notice the different parameters being varied horizontally. On the l.h.s. ($0 \le \beta < 1$, stationarity prevails), p varies. On the r.h.s. ($1 < \beta \le 2$, stationary increments prevail), H varies.

$q = 2$. From Eqs. (8) and (10), we find

$$\beta = \zeta_2 + 1 = \min\{2H, 1\} + 1 > 1(H > 0)\,. \tag{12a}$$

In its multifractal incarnation at $H = 0$, the model is stationary so Eq. (8), hence [Eq. 12(a)], no longer apply. From Eqs. (2)–(5) we find $\beta = D_2$ (the "correlation" dimension for a process developing in one spatial dimension). In this case Eq. (6) yields $D_2 = 2 - \ln_2[4p^2 + 4(1-p)^2]$ hence

$$\beta = -\ln_2[1 - 2p(1-p)] < 1(H = 0)\,. \tag{12b}$$

Both theoretical spectral exponents [Eqs. 12(a) and (b)] are presented in Fig. 2. Notice that the singular limit of the bounded model ($H \to 0$, $p > 0$) and the weak variability limit of the unbounded model ($p \to 1/2$, $H = 0$) agree ($\beta \to 1^{\pm}$). "Flicker" noise ($\beta = 1$) lies exactly at the boundary between stationary scaling processes ($\beta < 1$) and those with stationary increments ($\beta > 1$). Further arguments for this delineation are provided elsewhere.[1]

$q = q_c$. This is the highest order moment where the ζ_q's of bounded models with smoothing exponent H coincide with those of fBm's having an identical parameter; namely, $q_c = 1/H$. Indeed, for $q > q_c$, we have $\zeta_q = 1$ for the bounded model [Eq. (10)] and $\zeta_q = qH$ for fBm. In other words, bounded models and fBm's are indistinguishable by structure functions alone for moments of order $q \le q_c$. In the framework of multifractals and singularity analysis, the so-called thermodynamical formalism[18] allows us to interpret a discontinuity in the (first) derivative of an exponent function like ζ_q as a (first order) phase transition. In the present model, this qualitative change in statistical behavior for large q (low "temperature", $1/q$) is traceable to the boundedness and related large scale stationarity. This characteristic feature of the model is further discussed in Ref. 19.

5. SUMMARY AND CONCLUSIONS

We study a variant of the random multiplicative cascade model where the weights go to 1 as the cascade proceeds. The resulting field then has upper and lower bounds. We studied in more detail the one-dimensional case where the weights are $1 \pm (1-2p)r^H$ $(0 \le p \le 1/2$, $0 \le H \le \infty)$ at scale $r = 2^{-n}$ after n cascade steps, relative to the unit outer scale. Due to the boundedness, the D_q all converge to one (lack of intermittency) with increasing n. The structure function exponents are more interesting, $H_q = \min\{H, 1/q\}$ in the limit

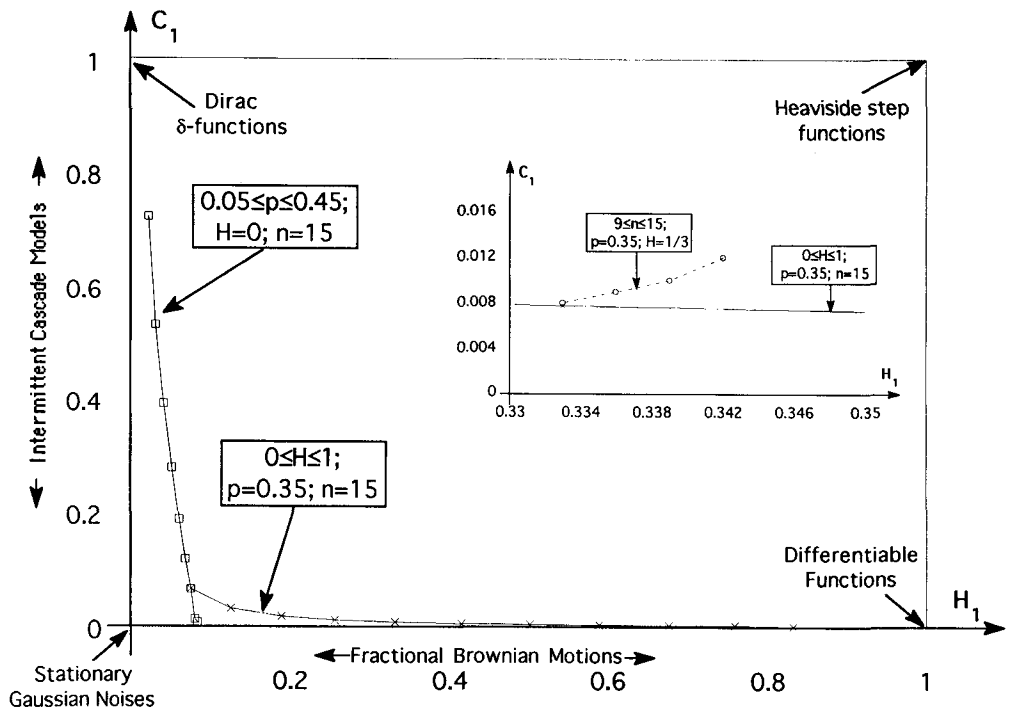

Fig. 3 Bounded and unbounded cascade models in the "multifractal $q = 1$ plane". Vertically, we have more-and-more intermittency, horizontally less-and-less stationarity. All four corners of the natural domain $(H_1, C_1) \in [0, 1] \otimes [0, 1]$ are occupied by four well-known cases, each of which is reached by special values of p and H (at $n = \infty$):

C_1	H_1	Description	p	H	D_g	β
0	0	weakly variable scaling stationary noises	$\approx \frac{1}{2}$	0	2	< 1
0	1	almost everywhere differentiable functions	$< \frac{1}{2}$	≥ 1	1	≥ 3
1	0	Dirac δ-functions	0	0	2	0
1	1	Heaviside step functions	$< \frac{1}{2}$	∞	1	2

For $n < \infty$, we find the model to navigate very near the axes: vertically if $H = 0$ (variable p), and horizontally if $H > 0$ (any p). The distance from the axes decreases with increasing n so we are simply dealing with finite size effects, essentially "spurious" intermittency or smoothness exponents (see inset).

$n = \infty$. In essence, by "turning on" the smoothing parameter H we have gone from a stochastically discontinuous multi-singular model to a bounded continuous multi-affine one. We further focused on three special statistical moments. The first moment ($q = 1$) is related to the graph dimension, the second ($q = 2$) is connected with the energy spectrum. The third ($q = 1/H$) is the critical order beyond which our multiplicative bounded cascade model, which is multi-scaling (in the sense of multi-affinity), can be distinguished from the associated additive model (fractional Brownian motion), which exhibits simple scaling.

We can summarize many of our findings with Fig. 3 where we represent the position of our $\{p, H; n\}$-models in what can be called the "$q = 1$ multifractal plane". The axes are both codimensions: the information codimension $C_1 = 1 - D_1$ (a direct measure of intermittency) vertically, and horizontally the graph codimension $H_1 = 2 - D_g$ (a direct measure of smoothness and non-stationarity). Such (H_1, C_1)-plots provide us with a powerful data-analysis tool.[1,20] To the best of our knowledge, only a handful of specific multi-affine models have been proposed.[7,21,22] We have developed a new one that is general enough to illustrate the key differences between scaling processes that are stationary and multiplicative, non-stationary and additive.

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