

# Monte Carlo Method and Transport Equation in Plant Canopies

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*Plant canopy reflectance is calculated using the governing equation for photon transport. The integral equation of transfer is solved by the Monte Carlo method. The main emphasis is on statistical estimation and simulation of the Markov chain. The leaf dimensions are taken into account in obtaining the hot-spot effect of the canopy. Finally, numerical results for transport equation obtained by the Monte Carlo method are compared with those obtained by using the method of discrete ordinates and the geometrical Monte Carlo model.*

## INTRODUCTION

For remote sensing of vegetation a number of canopy reflectance models have been developed during recent decades. Goel (1988) has classified the different ways of modeling the radiation regime in vegetation canopies into four categories: geometrical models, turbid medium models, hybrid

models, and computer simulation models. In the present paper, we study the models of the second category based on the radiative transfer theory and the statistical Monte Carlo models of the fourth category. Here the models of Shultis and Myneni (1988) and Ross and Marshak (1988) may be considered as typical representatives of these approaches.

Turbid medium models are most effective in dense and horizontally uniform canopies. However, the introduction of vertical and horizontal heterogeneities, row grown plants, leaf dimensions, effective distance between leaves, etc. in the model leads to significant computational difficulties. However, it is not difficult to consider these geometrical parameters using the Monte Carlo model. On the other hand, the inverse procedure or estimation of the canopy parameters from reflectance data for the statistical models of the Ross-Marshak type is rather complicated. In the framework of such an approach we cannot calculate the derivatives with respect to unknown geometrical parameters along the simulated photon trajectories (Marshak, 1987).

In the present paper we propose to unite these two approaches. We suggest that the radiation

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regime in the vegetation canopy should be described by the transport equation, namely, by the integral transport equation with inclusion of all the important geometrical parameters. Besides, we propose to solve this equation by the Monte Carlo method. Thus, we can complement the technique of solving the direct and inverse problems in atmospheric physics (Mikhailov, 1974) by the Monte Carlo method by applying these methods to the plant canopies.

For simplicity, we consider only homogeneous plant canopies and describe the main points of the Monte Carlo procedure. The first two sections are devoted to the integro-differential and integral equations in plate turbid medium. In this approach, the elements of vegetation are treated as small absorbing and scattering particles with given optical properties, distributed randomly in horizontal layers and oriented in given directions (Goel, 1988). The next two sections represent the method of solving the integral equation by the Monte Carlo method. Then, the hot-spot effect by considering finite dimensional scattering centers is included in our model. Numerical comparisons with other models are provided in the section, Numerical Results.

## TRANSPORT EQUATION IN SLAB-GEOMETRY FOR PLANT CANOPIES

We assume that the angular distribution of leaves is independent of depth and that it is defined by the leaf-normal probability density  $(2\pi)^{-1}g(\mathbf{\Omega}_L)$ , where  $\mathbf{\Omega}_L \sim (\theta_L, \phi_L)$ , is a leaf normal directed away from its upper surface into a unit solid angle about  $\mathbf{\Omega}_L$ . Then (Ross, 1981) the function  $G$  (dimensionless),

$$G(\mathbf{\Omega}) = \frac{1}{2\pi} \int_{2\pi^+} g_L(\mathbf{\Omega}_L) |\mathbf{\Omega} \cdot \mathbf{\Omega}_L| d\mathbf{\Omega}_L, \quad (1)$$

is the mean projection of leaf-normals on direction  $\mathbf{\Omega}$ . The unit vector  $\mathbf{\Omega} \sim (\mu, \phi)$  has an azimuthal angle  $\phi$  and a polar angle  $\theta = \cos^{-1}(\mu)$  with respect to the outward normal (opposite the  $z$ -axis which is directed down into the canopy).

We consider a flat, horizontal leaf canopy of depth  $T$ , which is illuminated from above by a direct monodirectional solar component in direc-

tion  $\mathbf{\Omega}_0 \sim (\mu_0, \phi_0)$  ( $\mu_0 < 0$ ) with intensity  $I_0$ . At the ground interface, it is assumed that a fraction  $r_s$  of the energy reaching the ground through the canopy is reradiated isotropically back into the canopy. The radiative regime in such a canopy is described by the transfer boundary-value problem (Shultis and Myneni, 1988)

$$\begin{aligned} -\mu \frac{\partial}{\partial \tau} I(\tau, \mu) + G(\mathbf{\Omega}) I(\tau, \mu) \\ = \omega_L \int_{4\pi} P(\mathbf{\Omega}' \rightarrow \mathbf{\Omega}) G(\mathbf{\Omega}') I(\tau, \mathbf{\Omega}') d\mathbf{\Omega}', \end{aligned} \quad 0 < \tau < H,$$

$$I(0, \mathbf{\Omega}) = I_0 \delta(\mathbf{\Omega} - \mathbf{\Omega}_0), \quad \mu < 0,$$

$$I(H, \mathbf{\Omega}) = \frac{r_s}{\pi} \int_{2\pi^-} |\mu'| I(H, \mathbf{\Omega}') d\mathbf{\Omega}', \quad \mu > 0. \quad (2)$$

Here,  $4\pi$ ,  $2\pi^+$ , and  $2\pi^-$  (introduced later) denote that the respective integrals are over the whole unit sphere, upper, and lower hemisphere. The cumulative leaf area index  $\tau(z)$  is defined as

$$\tau(z) = \int_0^z u_L(z) dz,$$

where  $u_L(z)$  is the total one-sided leaf area per unit volume of the canopy. The optical depth in the plant canopy depends on the direction of photon travel and is defined as  $\tau(z)G(\mathbf{\Omega})$ . Here  $0 \leq z \leq T$  is the geometrical depth of the canopy and

$$H = \int_0^T u_L(z) dz$$

is a leaf area index. The area scattering transfer function is defined as

$$\begin{aligned} P(\mathbf{\Omega}' \rightarrow \mathbf{\Omega}) = \frac{1}{2\pi} \int_{2\pi^+} g_L(\mathbf{\Omega}_L) |\mathbf{\Omega}' \cdot \mathbf{\Omega}_L| \\ \times f(\mathbf{\Omega}_L; \mathbf{\Omega}' \rightarrow \mathbf{\Omega}) d\mathbf{\Omega}_L / G(\mathbf{\Omega}'). \end{aligned} \quad (3)$$

The leaf phase function  $f(\mathbf{\Omega}_L; \mathbf{\Omega}' \rightarrow \mathbf{\Omega})$  is normalized to the single-scatter leaf albedo  $\omega_L$ , i.e.,

$$\int_{4\pi} f(\mathbf{\Omega}_L; \mathbf{\Omega}' \rightarrow \mathbf{\Omega}) d\mathbf{\Omega} = \omega_L.$$

It leads to the same equality for the transfer function  $P(\mathbf{\Omega}' \rightarrow \mathbf{\Omega})$ , i.e.,

$$\int_{4\pi} P(\mathbf{\Omega}' \rightarrow \mathbf{\Omega}) d\mathbf{\Omega} = \omega_L.$$

Next we obtain the boundary-value problem

$$\begin{aligned}
 & -\frac{\mu}{G(\boldsymbol{\Omega})} \frac{\partial}{\partial \tau} J(\tau, \boldsymbol{\Omega}) + J(\tau, \boldsymbol{\Omega}) \\
 & = \int_{-4\pi} P(\boldsymbol{\Omega}' \rightarrow \boldsymbol{\Omega}) J(\tau, \boldsymbol{\Omega}') d\boldsymbol{\Omega}', \quad (4) \\
 & J(0, \boldsymbol{\Omega}) = G(\boldsymbol{\Omega}) I(0, \boldsymbol{\Omega}), \quad \mu < 0, \\
 & J(H, \boldsymbol{\Omega}) = G(\boldsymbol{\Omega}) I(H, \boldsymbol{\Omega}), \quad \mu > 0
 \end{aligned}$$

by denoting the product  $I(\tau, \boldsymbol{\Omega})G(\boldsymbol{\Omega})$  in (2) by  $J(\tau, \boldsymbol{\Omega})$ .

**THE INTEGRAL EQUATION**

We derive the integral equation for the plant canopy neglecting reflectance from the ground. Denoting the integral term in (4) by  $y(\tau, \boldsymbol{\Omega})$  and solving the boundary-value problem, we obtain

$$\begin{aligned}
 & J(\tau, \boldsymbol{\Omega}) \\
 & = \begin{cases} \frac{G(\boldsymbol{\Omega})}{|\mu|} \int_0^\tau y(\tau', \boldsymbol{\Omega}) \exp\left[-\frac{G(\boldsymbol{\Omega})}{|\mu|}(\tau - \tau')\right] d\tau' \\ \quad + J(0, \boldsymbol{\Omega}) \exp\left[-\frac{G(\boldsymbol{\Omega})}{|\mu|}\tau\right], & \mu < 0, \\ \frac{G(\boldsymbol{\Omega})}{\mu} \int_\tau^H y(\tau', \boldsymbol{\Omega}) \exp\left[-\frac{G(\boldsymbol{\Omega})}{\mu}(\tau' - \tau)\right] d\tau', & \mu < 0, \\ y(\tau, \boldsymbol{\Omega}), & \mu > 0. \end{cases}
 \end{aligned}$$

Substituting the right-hand side of the above equation in the boundary-value problem (4) for  $y$ , the integral equation can be written as

$$\begin{aligned}
 J(\tau, \boldsymbol{\Omega}) & = \int_{-4\pi} \int_0^H k[(\tau', \boldsymbol{\Omega}') \rightarrow (\tau, \boldsymbol{\Omega})] \\
 & \quad \times J(\tau', \boldsymbol{\Omega}') d\tau' d\boldsymbol{\Omega}' + Q(\tau, \boldsymbol{\Omega}), \quad (5)
 \end{aligned}$$

where

$$k(x' \rightarrow x) = \begin{cases} k_-(x' \rightarrow x), & \mu < 0, \\ k_+(x' \rightarrow x), & \mu > 0, \end{cases} \quad (6)$$

and  $x \sim (\tau, \boldsymbol{\Omega})$  is a point in the phase space  $X$ .

Here,

$$\begin{aligned}
 & k_-(x' \rightarrow x) \\
 & = \begin{cases} P(\boldsymbol{\Omega}' \rightarrow \boldsymbol{\Omega}) \frac{G(\boldsymbol{\Omega})}{|\mu|} \exp\left[-\frac{G(\boldsymbol{\Omega})}{|\mu|}(\tau - \tau')\right], \\ \quad 0 \leq \tau' \leq \tau, \\ 0, & \tau < \tau' \leq H, \end{cases} \quad (7)
 \end{aligned}$$

$$\begin{aligned}
 & k_+(x' \rightarrow x) \\
 & = \begin{cases} 0, & 0 \leq \tau' \leq \tau, \\ P(\boldsymbol{\Omega}' \rightarrow \boldsymbol{\Omega}) \frac{G(\boldsymbol{\Omega})}{\mu} \exp\left[-\frac{G(\boldsymbol{\Omega})}{\mu}(\tau' - \tau)\right], \\ \quad \tau < \tau' \leq H, \end{cases} \quad (8)
 \end{aligned}$$

and

$$Q(\tau, \boldsymbol{\Omega}) = J(0, \boldsymbol{\Omega}) \exp\left[-\frac{G(\boldsymbol{\Omega})}{|\mu|}\tau\right], \quad \mu < 0.$$

Note that integral equation (5) can be derived from physical motivations as well (Pomraning, 1973).

*Remark:* In case of nonzero soil albedo ( $r_s \neq 0$ ), the kernel  $k_+$  for  $\tau \leq \tau' \leq H$  and the source function  $Q$  for  $\mu > 0$  can be expressed as

$$\begin{aligned}
 & k_+(x' \rightarrow x) \\
 & = P(\boldsymbol{\Omega}' \rightarrow \boldsymbol{\Omega}) \frac{G(\boldsymbol{\Omega})}{\mu} \exp\left[-\frac{G(\boldsymbol{\Omega})}{\mu}(\tau' - \tau)\right] \\
 & \quad + \frac{r_s}{\pi} G(\boldsymbol{\Omega}) \exp\left[-\frac{G(\boldsymbol{\Omega})}{\mu}(H - \tau)\right] \\
 & \quad \times \int_{2\pi} P(\boldsymbol{\Omega}' \rightarrow \boldsymbol{\Omega}'') \\
 & \quad \times \exp\left[-\frac{G(\boldsymbol{\Omega}'')}{|\mu''|}(H - \tau')\right] d\boldsymbol{\Omega}'',
 \end{aligned}$$

$$\begin{aligned}
 Q(\tau, \boldsymbol{\Omega}) & = I_0 G(\boldsymbol{\Omega}) \frac{r_s |\mu_0|}{\pi} \exp\left[-\frac{G(\boldsymbol{\Omega}_0)}{|\mu_0|}\tau\right] \\
 & \quad \times \exp\left[-\frac{G(\boldsymbol{\Omega})}{\mu}(H - \tau)\right].
 \end{aligned}$$

**SOLUTION OF THE INTEGRAL EQUATION BY THE MONTE CARLO METHOD**

We rewrite the integral equation (5) in operator notation as

$$J = KJ + Q, \quad (9)$$

where the kernel  $k(x' \rightarrow x)$  of the integral operator  $K$  is defined by (6)–(8). The kernel  $k(x' \rightarrow x)$  is a probability density of interaction at the point  $x$  of a phase space provided the previous interaction occurred at the point  $x'$ . Here  $x \sim (\tau, \mathbf{\Omega})$ , where  $\tau$  is a cumulative leaf area index at the point of interaction and  $\mathbf{\Omega}$  is the direction of the photon just before interaction;  $J(\tau, \mathbf{\Omega})$  is the phase interaction density;  $Q(\tau, \mathbf{\Omega})/|\mu|$  is the density of the first collision photons of intensity  $I(0, \mathbf{\Omega})$  in the direction  $\mathbf{\Omega}$ .

It is known that if the condition  $\|K\| < 1$  holds, Eq. (9) has a unique solution that can be represented in Neuman's series (Riesz and Sz.-Nagy, 1972)

$$J = Q + KQ + K^2Q + \dots \quad (10)$$

For remote sensing purposes, it is sufficient to know the value of  $J(0, \mathbf{\Omega}^*)$ ,  $\mathbf{\Omega}^* \sim (\mu^*, \phi^*)$ ,  $\mu^* > 0$ , i.e., the reflection from the upper surface in the direction  $\mathbf{\Omega}^*$ . Then,

$$J(0, \mathbf{\Omega}^*) = (J, \delta_{x^*}) = \int_X J(x) \delta_{x^*}(x) dx,$$

where  $\delta_{x^*}$  is the Dirac delta function and  $x^* \sim (0, \mathbf{\Omega}^*)$ . Taking into account (10), we have

$$\begin{aligned} J(0, \mathbf{\Omega}^*) &= (Q, \delta_{x^*}) + (KQ, \delta_{x^*}) + (K^2Q, \delta_{x^*}) + \dots \\ &= (Q, K^* \delta_{x^*}) + (KQ, K^* \delta_{x^*}) \\ &\quad + (K^2Q, K^* \delta_{x^*}) + \dots \end{aligned}$$

We have used the equality  $(Q, \delta_{x^*}) = 0$  since the incident solar and the reflected beams are in opposing hemispheres. The operator  $K^*$  is the operator adjoint to  $K$  and

$$(K^* \eta)(x) = \int_X k(x \rightarrow x') \eta(x') dx'$$

Then,

$$\begin{aligned} (K^* \delta_{x^*})(\tau, \mathbf{\Omega}) &= \int_{4\pi} \int_0^H k[(\tau, \mathbf{\Omega}) \rightarrow (\tau', \mathbf{\Omega}')] \\ &\quad \times \delta_{x^*}(\tau', \mathbf{\Omega}') d\tau' d\mathbf{\Omega}' \\ &= k[(\tau, \mathbf{\Omega}) \rightarrow (0, \mathbf{\Omega}^*)] \\ &= \Psi(\tau, \mathbf{\Omega}) G(\mathbf{\Omega}^*), \end{aligned}$$

where the contribution function  $\Psi(\tau, \mathbf{\Omega})$  is

$$\begin{aligned} \Psi(\tau, \mathbf{\Omega}) &= P(\mathbf{\Omega} \rightarrow \mathbf{\Omega}^*) \frac{1}{\mu^*} \exp\left[-\frac{G(\mathbf{\Omega}^*)}{\mu^*} \tau\right], \\ &\mu^* > 0. \end{aligned}$$

Hence, it follows that

$$\begin{aligned} I(0, \mathbf{\Omega}^*) &= J(0, \mathbf{\Omega}^*) / G(\mathbf{\Omega}^*) \\ &= (Q, \Psi) + (KQ, \Psi) + (K^2Q, \Psi) + \dots \end{aligned} \quad (11)$$

Here,  $\mathbf{\Omega}^*$  is the view direction at  $\tau = 0$  (the upper surface). In case of reflection from the soil, the contribution function  $\Psi$  is

$$\begin{aligned} \Psi_s(\tau, \mathbf{\Omega}) &= \frac{r_s}{\pi} \frac{|\mu|}{G(\mathbf{\Omega})} \exp\left[-\frac{G(\mathbf{\Omega}^*)}{\mu^*} \tau\right] \delta(\tau - H), \\ &\mu^* > 0. \end{aligned}$$

The  $i$ th term in the right-hand part of (1) is the contribution of  $i$ th order scattered photons to the estimation of the canopy reflectance. Expansion (11) corresponds to the algorithm of the Monte Carlo method; namely, according to the two probability densities (probability density of the length of photon free path in direction  $\mathbf{\Omega}$  and the probability density of the scattering in a solid angle about  $\mathbf{\Omega}$ ) in the kernel  $k(x' \rightarrow x)$ , the photon trajectory  $x_0'' \rightarrow x_1'' \rightarrow x_2'' \rightarrow \dots \rightarrow x_m''$  is simulated. Here,  $x_i''$  are the points of  $i$ th interactions for the  $n$ th photon in the phase-space  $X$  and  $m$  is the random number of the last interaction. After each interaction at the point  $x_i''$  the contribution  $[W_i'' \Psi(x_i'')]$  is included in the statistical estimation  $I$  and

$$I(0, \mathbf{\Omega}^*) \approx \frac{1}{N} \sum_{n=1}^N \sum_{i=0}^m W_i'' \Psi(x_i''), \quad (12)$$

where  $N$  is the total number of photons,  $W_i''$  is the "weight" of the  $n$ th photon after  $i$ th interaction.

For obtaining the bidirectional reflectance factor  $R(\mathbf{\Omega}_0, \mathbf{\Omega}^*)$ , it is sufficient to multiply  $I(0, \mathbf{\Omega}^*)$  by  $\pi / I_0 |\mu_0|$ , i.e.,

$$R(\mathbf{\Omega}_0, \mathbf{\Omega}^*) = \pi I(0, \mathbf{\Omega}^*) / I_0 |\mu_0|.$$

### SIMULATION OF THE MARKOV CHAIN

What we now need for the final solution of the transport equation is only the simulation of the Markov chain.

1. *Simulation of photon mean free path.* According to the standard method of simulation of a continuous random value, the mean optical length of the photon free path  $\bar{\tau}$  can be found from the equation (Mikhailov, 1974)

$$F(\bar{\tau}) = \alpha, \quad (13)$$

where  $\alpha$  is a random number uniformly distributed in  $(0, 1)$  and

$$F(t) = 1 - \exp(-t) \quad [t = \tau G(\mathbf{\Omega}) / |\mu|] \quad (14)$$

is a function of the distribution of  $\bar{\tau}$ . From (13) and (14) it follows that

$$\bar{\tau} = -|\mu| \ln \alpha / G(\mathbf{\Omega}).$$

2. *Simulation of the scattering direction.* Let  $\mathbf{\Omega}'$  be a direction of photon travel before interaction. We simulate the direction  $\mathbf{\Omega}$  after interaction as follows.

Note that the probability density of a simulation should satisfy the following two conditions: it should be convenient for simulation and universal. The condition of "universality" means that the density used in the algorithm should not change with real distribution. In the opposite case, one should look for a new simulation density for each set of initial data. The proposed Monte Carlo method uses some general and convenient density. The bias of statistical estimation in this case is compensated by the product of the photon "weight" with an appropriate factor at each act of interaction. The "weight" should be multiplied by the contribution function in the estimation (12).

Thus, we represent function  $P(\mathbf{\Omega}' \rightarrow \mathbf{\Omega})$  by the superposition of two densities, viz.,

$$P(\mathbf{\Omega}' \rightarrow \mathbf{\Omega}) = \int_{2\pi} \frac{g_L(\mathbf{\Omega}_L)}{2\pi} \frac{|\mathbf{\Omega}' \cdot \mathbf{\Omega}_L|}{G(\mathbf{\Omega}')} \\ \times f(\mathbf{\Omega}_L; \mathbf{\Omega}' \rightarrow \mathbf{\Omega}) d\mathbf{\Omega}_L.$$

Then, the simulation of the direction  $\mathbf{\Omega}$  is carried out in the following way: The outward normal  $\mathbf{\Omega}_L$  is simulated according to the density

$$p(\mathbf{\Omega}_L) = \frac{g_L(\mathbf{\Omega}_L)}{2\pi} \frac{|\mathbf{\Omega}' \cdot \mathbf{\Omega}_L|}{G(\mathbf{\Omega}')}.$$

Then, knowing  $\mathbf{\Omega}_L$ , the direction  $\mathbf{\Omega}$  is simulated according to the density  $f(\mathbf{\Omega}_L; \mathbf{\Omega}' \rightarrow \mathbf{\Omega})$ . However, the density  $p(\mathbf{\Omega}_L)$  does not satisfy the two above conditions. As a matter of fact there are no simple simulation formulae available and the probability density of leaf normals  $g_L(\mathbf{\Omega}_L)/2\pi$  is generally a polyparameterical function.

Assuming that leaf normals are uniformly distributed with respect to azimuth  $\phi$ , their distribution with respect to  $\mu$  is represented by the three-parameter family (Bunnik, 1978), the density

$g_L(\mathbf{\Omega}_L)/2\pi$  can be written as

$$\frac{g_L(\mathbf{\Omega}_L)}{2\pi} = \frac{2}{\pi} \frac{1}{\sqrt{1-\mu_L^2}} \frac{1}{2\pi} g^*(\mu_L),$$

where

$$g^*(\mu) = a + b(2\mu^2 - 1) + c(8\mu^4 - 8\mu^2 + 1), \\ a, b, c = \text{const.} \quad (15)$$

We shall consider the function

$$(2/\pi)(1-\mu^2)^{-0.5}(1/2\pi)$$

as a new probability density for the direction  $\mathbf{\Omega}_L \sim (\mu_L, \phi_L)$ , where  $1/2\pi$  is the density with respect to  $\phi_L$  and  $(2/\pi)(1-\mu^2)^{-0.5}$  is the density with respect to  $\mu_L$ . The simulation formulae are very simple:

$$\phi_L = 2\pi\alpha, \quad \mu_L = \sin(\pi\beta/2),$$

where  $\alpha$  and  $\beta$  are uniformly distributed in  $(0, 1)$ . So, the probability density with respect to  $\mathbf{\Omega}_L$  is both universal (independent of parameters  $a, b, c$ ) and convenient.

To simulate a more general distribution function of leaf normals with respect to  $\phi$ , it is sufficient to change the density  $1/2\pi$  to the appropriate one. Assuming independence in distribution about polar and azimuthal angles of leaf normals, one should to change the simulation formulae for  $\phi_L$  only. Thus, separate the above density from  $p(\mathbf{\Omega}_L)$ . We have

$$p(\mathbf{\Omega}_L) = \left[ \frac{2}{\pi} \frac{1}{\sqrt{1-\mu_L^2}} \frac{1}{2\pi} \right] \left[ g^*(\mu_L) \frac{|\mathbf{\Omega}' \cdot \mathbf{\Omega}_L|}{G(\mathbf{\Omega}')} \right].$$

The first factor will be the new density for  $\mathbf{\Omega}_L$  and the second factor is produced by the "weight" of the photon at the following interaction. Note that the factorization can be carried out in a different way as well. However, our approach has the advantage that the unbounded factor,  $(1-\mu_L^2)^{-0.5}$ , is included in the density and excluded from the "weight." It allows us to obtain a bounded estimate. The unboundedness of the estimate leads to great errors.

Knowing  $\mathbf{\Omega}_L$ , the simulation of the direction  $\mathbf{\Omega}$  is performed accordingly to the leaf scattering probability density  $f(\mathbf{\Omega}_L; \mathbf{\Omega}' \rightarrow \mathbf{\Omega})$ . For a "bi-Lambertian case" (Shultis and Myneni, 1988) it

can be written as

$$f(\mathbf{\Omega}_L; \mathbf{\Omega}' \rightarrow \mathbf{\Omega}) = \begin{cases} r_L |\mathbf{\Omega} \cdot \mathbf{\Omega}_L| / \pi, & (\mathbf{\Omega} \cdot \mathbf{\Omega}_L)(\mathbf{\Omega}' \cdot \mathbf{\Omega}_L) < 0, \\ t_L |\mathbf{\Omega} \cdot \mathbf{\Omega}_L| / \pi, & (\mathbf{\Omega} \cdot \mathbf{\Omega}_L)(\mathbf{\Omega}' \cdot \mathbf{\Omega}_L) > 0, \end{cases}$$

where  $r_L$  and  $t_L$  are the leaf hemispherical reflectance and transmittance, respectively, and a simulation can be obtained successively by means of the formulæ

$$\phi = 2\pi\alpha, \quad \mu = \delta\sqrt{\beta},$$

where  $\mu$  and  $\phi$  are the polar coordinates of the vector  $\mathbf{\Omega}$  with axis  $\mathbf{\Omega}_L$  and

$$\delta = \begin{cases} +1, & \left\{ (\mathbf{\Omega}' \cdot \mathbf{\Omega}_L) > 0 \right\}, \& \left[ \gamma > \frac{r_L}{r_L + t_L} \right] \\ \text{or} & \left\{ (\mathbf{\Omega}' \cdot \mathbf{\Omega}_L) < 0 \right\} \& \left[ \gamma < \frac{r_L}{r_L + t_L} \right], \\ -1, & \text{otherwise.} \end{cases}$$

Here  $\alpha$ ,  $\beta$ , and  $\gamma$  are independent random values uniformly distributed in (0, 1).

3. *Calculation of the contribution function.* As a contribution function we should not treat the accurate meaning of  $\Psi(\tau, \mathbf{\Omega})$  (that is very difficult to calculate after each interaction), but we require its estimation randomized with respect to  $\mathbf{\Omega}_L$ , namely,  $[f(\mathbf{\Omega}_L; \mathbf{\Omega}' \rightarrow \mathbf{\Omega}^*) \exp(-\tau^*) / \mu^*]$ . Here,  $\mathbf{\Omega}_L$  is a leaf normal simulated after interaction;  $\tau^*$  is the optical depth from the point of interaction to the receiver in the direction  $\mathbf{\Omega}^*$ .

4. *The general algorithm of the Markov chain simulation.* There are the following steps:

- Get the initial position of the photon.
- Simulate the length of photon free path (see 1) and calculate the  $\tau$ .
- Simulate the leaf normal  $\mathbf{\Omega}_L$ .
- Evaluate the "weight" of a photon with the appropriate factor (see 2) and leaf albedo  $\omega_L$ .
- Calculate of the contribution in the statistical estimate (12).
- Simulate the new photon direction  $\mathbf{\Omega}$ .

This procedure is continued until the photon escapes the canopy ( $\tau < 0$ ) or is absorbed. Then a new trajectory begins.

## CONSIDERATION OF THE HOT-SPOT EFFECT

The above model satisfactorily approximates canopies in which the size of phytoelements is small compared with the height of the canopy. The consideration of mean leaf size in the canopy leads to a shading effect called the hot-spot effect (Kuusk, 1985; Gerstl et al., 1986; Ross and Marshak, 1988). In this case the extinction coefficient depends at least on both the initial direction  $\mathbf{\Omega}'$  and scattered photon direction  $\mathbf{\Omega}$ . Using the coefficient of mutual correlation of the indicator functions of gaps in directions  $\mathbf{\Omega}$  and  $\mathbf{\Omega}'$  (Nilson and Kuusk, 1989), a new extinction coefficient  $\sigma$  depending on the mean leaf dimension was obtained (Marshak, 1989). We assume that the parameter  $\kappa$  is defined as  $\kappa = l_L / T$ , where  $l_L$  is the length of mean chord of the leaf and  $T$  is the height of canopy (Kuusk, 1985; Nilson and Kuusk, 1989). In case of a homogeneous leaf canopy we have

$$\sigma_\kappa[(\tau', \mathbf{\Omega}') \rightarrow (\tau, \mathbf{\Omega})] = \begin{cases} G(\mathbf{\Omega}), & \mu\mu' > 0, \\ G(\mathbf{\Omega})h_\kappa(|\tau' - \tau|, \mathbf{\Omega}', \mathbf{\Omega}), & \mu\mu' < 0, \end{cases} \quad (16)$$

where

$$h_\kappa(t, \mathbf{\Omega}', \mathbf{\Omega}) = 1 - \left[ \frac{G(\mathbf{\Omega}')|\mu|}{G(\mathbf{\Omega})|\mu'|} \right]^{1/2} \times \exp \left[ - \frac{\Delta(\mathbf{\Omega}, \mathbf{\Omega}')t}{\kappa H} \right],$$

$$\Delta(\mathbf{\Omega}, \mathbf{\Omega}') = (\mu^{-2} + \mu'^{-2} + 2(\mathbf{\Omega}' \cdot \mathbf{\Omega}) / |\mu\mu'|)^{1/2}.$$

It can be seen that in case of back-scattering ( $\mathbf{\Omega} = -\mathbf{\Omega}'$ ) extinction is absent and  $\sigma_\kappa = 0$ . Hence, we use new extinction coefficient (16) instead of (1) for simulating the photon free path. For more details regarding the consideration of finite dimensional scatterers in transport problem, see the forthcoming paper of Myneni et al. (1990).

## NUMERICAL RESULTS

To illustrate the proposed Monte Carlo method, numerical results for the canopy bidirectional reflectance factor  $R(\mathbf{\Omega}_0, \mathbf{\Omega})$  are now presented and compared to those calculated by other methods.

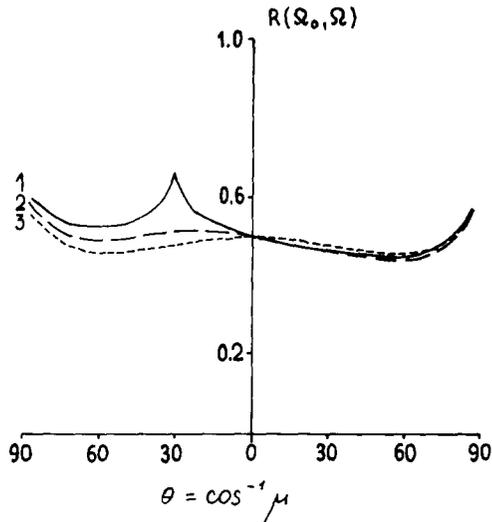


Figure 1. The bidirectional reflectance factor of the plant canopy  $R(\Omega_0, \Omega)$  for various vertical cross sections. 1)  $\phi = \phi_0$  and  $\phi = \phi_0 + 180^\circ$ ; 2)  $\phi = \phi_0 + 45^\circ$  and  $\phi = \phi_0 + 225^\circ$ ; 3)  $\phi = \phi_0 + 90^\circ$  and  $\phi = \phi_0 + 270^\circ$ . Here,  $r_L = t_L = 0.46$ ,  $r_s = 0.0$ ,  $H = 3.0$ ,  $\kappa = 0.08$ ,  $\theta_0 = 150^\circ$ .

In Figure 1 we show three vertical cross sections of the bidirectional reflectance factor to represent the planophile canopy [mainly horizontal leaves:  $a = 1$ ,  $b = 1$ ,  $c = 0$  in Eq. (15)] with  $\mu_0 = -\sqrt{3}/2$ . Symmetry with respect to the nadir occurs only for the cross sections  $\phi = \phi_0 + 90^\circ$  and  $\phi = \phi_0 + 270^\circ$ .

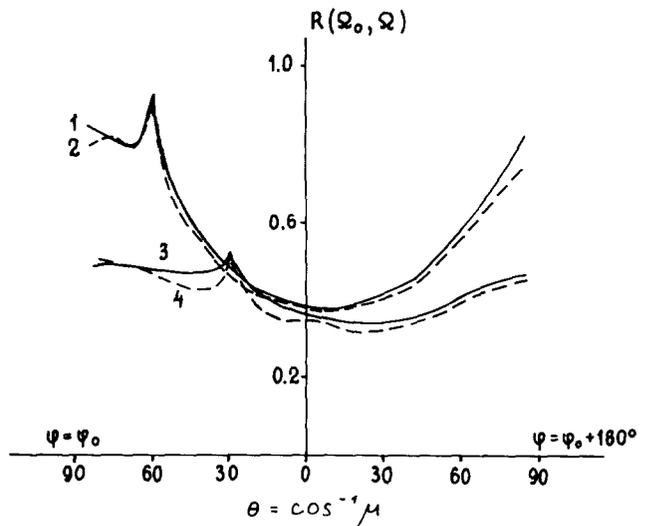
We now compare two Monte Carlo models: the above proposed model based on the transport equation (Model 1) and the geometrical Ross-Marshak model (Model 2) (Ross and Marshak, 1988; 1989). Figure 2 illustrates the reflectance factors  $R(\Omega_0, \Omega)$  in the near-infrared spectral region for the erectophile canopy [mainly vertical leaves:  $a = 1$ ,  $b = -1$ ,  $c = 0$  in Eq. (15)]. We show four curves. Curves 1 and 2 refer to  $\mu_0 = -\sqrt{3}/2$  and curves 3 and 4 illustrate  $\mu_0 = -0.5$ . There is good agreement between the two models. However, the bidirectional reflectance for Model 1 (curves 1 and 3) is greater than the bidirectional reflectance for Model 2 (curves 2 and 4). In Model 2, the phytoelements are distributed more regularly and the probability to view the dark soil is greater than for Model 1 with randomly distributed leaves. The same effect has been observed in calculations using the discrete ordinates method instead of Model 1 by Marshak (1989).

Figure 3 demonstrates the influence of the dimensions on the bidirectional reflectance factor  $R(\Omega_0, \Omega)$  in the near-infrared region. Curve 1

refers to the solution of the transport boundary-value problem (2). Curves 2 and 3 represent the solution of the transport equation with the new extinction coefficient  $\sigma_\kappa$  taking into account the leaf dimensions [see (16)]. Curve 2 illustrates the influence of leaf sizes on first-order scattering and curve 3 considers leaf sizes for first-order scattering as well as for multiply-scattered photons. The difference between curves 2 and 3 denotes the influence of leaf dimensions on multiply scattered photons. For the planophile canopy in the principal plane [Fig. 3a], there is a constant difference for all view directions. However, for the erectophile leaves the difference increases significantly in large view directions [Fig. 3b]. It can be explained by a significant hot-spot brightness in an erectophile canopy along near-horizontal view directions. Note that curves 2 for both the planophile and the erectophile leaves coincide with those calculated by the discrete ordinates method with separation of first-order scattering (Marshak, 1989).

Finally, we give some quantitative measure of computer time. It is always desirable to use statistical simulation methods such as Monte Carlo. If we compare the method presented above with the discrete ordinates method (Marshak, 1989) and the geometrical Monte Carlo method (Ross and Marshak, 1989), we can order them for the following way: the discrete ordinates method, the present

Figure 2. Comparison of the bidirectional reflectance factors calculated by the model based on the transport equation [curves 1) and 3)] with those obtained by the geometrical Monte Carlo model [curves 2) and 4)]. Here,  $r_L = t_L = 0.46$ ,  $r_s = 0.2$ ,  $H = 3.0$ ,  $\kappa = 0.08$ ,  $\phi = \phi_0$ ,  $\theta_0 = 150^\circ$  [curves 1) and 2)],  $\theta_0 = 120^\circ$  [curves 3) and 4)]. The principal plane.



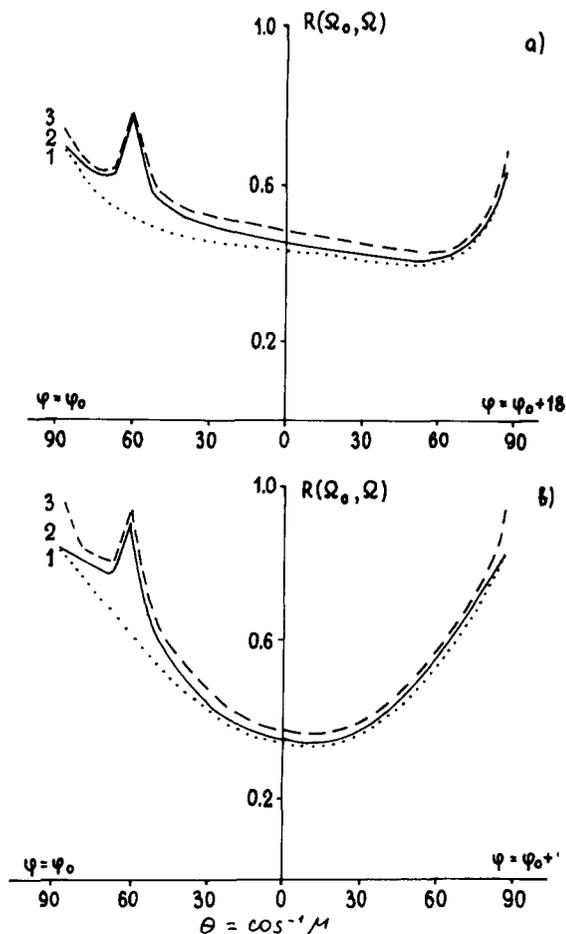


Figure 3. Consideration of the leaf dimensions for both the planophile canopy (a) and the erectophile canopy (b). 1)  $\kappa = 0.0$  (infinitesimally small leaves); 2)  $\kappa = 0.08$  for first-order scattering, only; 3)  $\kappa = 0.08$  for both first-order scattered photons and multiply-scattered photons. Here,  $r_L = t_L = 0.46$ ,  $r_s = 0.0$ ,  $H = 3.0$ ,  $\theta_0 = 120^\circ$ ,  $\phi = \phi_0$ . The principal plane.

Monte Carlo method for transport equation, and the geometrical Monte Carlo method. The computer time for each of these methods differs by a factor 2, i.e., the computer time for the discrete ordinates method (with 144 discrete directions) is two times less than that for the Monte Carlo method, for the same one-dimensional transport equation (about 1000 photon histories are needed to obtain mean square-root errors less than 5%) and four times less than the computer time needed for the geometrical Monte Carlo procedure which requires a more complicated model of the plant canopy. We should like to emphasize that these comparisons are valid only for the simplest case of a one-dimensional problem. The more complicated a medium, the less the advantage of the discrete

ordinate method and, naturally, for the most real leaf canopy, both Monte Carlo methods will be superior to the multidimensional discrete ordinates model. The main reason is that the Monte Carlo method does not require much more computer time for three-dimensional calculations than for one-dimensional problems.

## CONCLUSIONS

We have considered the Monte Carlo technique for the solution of the integral transport equation in a plate medium. It allows us to take into account an important geometrical parameter, the leaf dimension, that leads to the formation of the canopy hot-spot effect. The proposed model gives a possibility to supplement it by including both the structural and the optical parameters like specular reflection component in the scattering phase function. The proposed technique allows us also to estimate the contribution of the canopy hot-spot effect for the radiance of the multiply scattered photons (Fig. 3). Further development of the presented model involves generalization to the 3-D case and inversion of geometrical and optical parameters using data on canopy reflectance. The transport theory for a leaf canopy of finite dimensional scattering centers is described in the paper by Myneni, Marshak, and Knyazikhin (1990).

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